

**NILPOTENT SYMMETRY INVARIANCE IN QED  
WITH DIRAC FIELDS: SUPERFIELD FORMALISM****R. P. MALIK<sup>1</sup>***International School for Advanced Studies (SISSA),  
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**Abstract:** We provide the geometrical interpretation for the nilpotent Becchi-Rouet-Stora-Tyutin (BRST) and anti-BRST symmetry invariance of the Lagrangian density of a four ( $3 + 1$ )-dimensional (4D) interacting  $U(1)$  gauge theory within the framework of the superfield approach to BRST formalism. This interacting theory, where there is an explicit coupling between the  $U(1)$  gauge field and matter (Dirac) fields, is considered on a  $(4, 2)$ -dimensional supermanifold parametrized by the four spacetime variables  $x^\mu$  ( $\mu = 0, 1, 2, 3$ ) and a pair of Grassmannian variables  $\theta$  and  $\bar{\theta}$  (with  $\theta^2 = \bar{\theta}^2 = 0, \theta\bar{\theta} + \bar{\theta}\theta = 0$ ). We express the Lagrangian density in the language of the superfields of the theory and show that the (anti-)BRST invariance of the 4D Lagrangian density is equivalent to the translation of the super Lagrangian density along the Grassmannian direction(s) ( $\theta$  and/or  $\bar{\theta}$ ) of the  $(4, 2)$ -dimensional supermanifold such that the outcome of the above translation(s) is zero.

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# 1 Introduction

The usual superfield approach [1-8] to BRST formalism has been very successfully applied to the case of the 4D (non-)Abelian 1-form ( $A^{(1)} = dx^\mu A_\mu$ ) gauge theories. In this approach, one constructs a super curvature 2-form  $\tilde{F}^{(2)} = \tilde{d}\tilde{A}^{(1)} + i\tilde{A}^{(1)} \wedge \tilde{A}^{(1)}$  by exploiting the Maurer-Cartan equation in the language of the super 1-form gauge connection  $\tilde{A}^{(1)}$  and the super exterior derivative  $\tilde{d} = dx^\mu \partial_\mu + d\theta \partial_\theta + d\bar{\theta} \partial_{\bar{\theta}}$  (with  $\tilde{d}^2 = 0$ ) that are defined on a  $(4, 2)$ -dimensional supermanifold, parametrized by the superspace variable  $Z^M = (x^\mu, \theta, \bar{\theta})$ . This is subsequently equated to the ordinary 2-form curvature  $F^{(2)} = dA^{(1)} + iA^{(1)} \wedge A^{(1)}$ , defined on the 4D ordinary flat Minkowskian spacetime manifold that is parametrized by the ordinary spacetime variable  $x^\mu$  ( $\mu = 0, 1, 2, 3$ ). This restriction, popularly known as the horizontality condition, leads to the derivation of the nilpotent (anti-)BRST symmetry transformations for the gauge and (anti-)ghost fields of the (non-)Abelian 1-form gauge theories. In the above,  $\tilde{d}$  and  $\tilde{A}^{(1)}$  are the generalizations of the ordinary 4D  $d = dx^\mu \partial_\mu$  and  $A^{(1)}$  on the above supermanifold.

The key reasons behind the emergence of the nilpotent (anti-)BRST symmetry transformations for the gauge and (anti-)ghost fields, due to the above horizontality condition<sup>2</sup> (HC), are

(i) the nilpotency of the (super) exterior derivatives  $(\tilde{d})d$  which play an important role in the above HC, and

(ii) the super 1-form connection  $\tilde{A}^{(1)} = dZ^M \tilde{A}_M$  involves the vector superfield  $\tilde{A}_M$  that consists of the multiplet superfields  $(\mathcal{B}_\mu, \mathcal{F}, \bar{\mathcal{F}})$  which are the generalizations of the gauge and (anti-)ghost fields  $(A_\mu, C, \bar{C})$  (that are the basic fields of the 4D (non-)Abelian 1-form gauge theory).

The above equality (i.e.  $\tilde{F}^{(2)} = F^{(2)}$ ), due to the HC, implies that the ordinary curvature 2-form  $F^{(2)}$  remains unaffected by the presence of the Grassmannian coordinates  $\theta$  and  $\bar{\theta}$  (with  $\theta^2 = \bar{\theta}^2 = 0, \theta\bar{\theta} + \bar{\theta}\theta = 0$ ) of the superspace variable  $Z^M$ . The above restriction, however, does not shed any light on the derivation of the nilpotent (anti-)BRST symmetry transformations that are associated with the matter (e.g. Dirac, complex scalar, etc.) fields of an interacting 4D (non-)Abelian 1-form gauge theory.

In a recent set of papers [10-20], the above HC has been extended so as to derive the nilpotent (anti-)BRST symmetry transformations for the matter (or analogous) fields within the framework of the superfield approach to BRST formalism without spoiling the cute geometrical interpretations for the nilpotent (anti-)BRST symmetry transformations (and their corresponding generators) that emerge from the application of the HC alone. In fact, in a

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<sup>2</sup>This condition has been christened as the soul-flatness condition in [9].

set of three papers [18-20], we have been able to generalize the HC by a gauge invariant restriction (GIR) on the matter superfields (defined on the above (4, 2)-dimensional supermanifold) which enables us to derive the nilpotent (anti-)BRST symmetry transformations *together* for the gauge, (anti-)ghost and matter fields of a 4D interacting (non-)Abelian gauge theory in one stroke. In this *single* GIR on the matter superfields of the above supermanifold, the (super) covariant derivatives and their intimate connection with the (super) curvature 2-forms, play a very decisive role.

In the earlier works on the superfield formulation [1-20], the nilpotent (anti-)BRST symmetry invariance of the physical 4D Lagrangian density of the (non-)Abelian 1-form gauge theories has not yet been captured. In our previous endeavours [21-25], we have attempted to capture the nilpotent symmetry invariance of the 2D (non-)Abelian 1-form gauge theories (without any interaction with matter fields) within the framework of the superfield approach to BRST formalism. However, these theories are found to be topological in nature and they are endowed with the nilpotent (anti-)BRST as well as nilpotent *(anti-)co-BRST* symmetry transformations. The geometrical interpretations for the Lagrangian density and symmetric energy momentum tensor of this theory have been provided in these attempts. In our very recent paper [26], we have been able to provide the geometrical interpretation for the (anti-)BRST invariance of the 4D free (non-)Abelian 1-form gauge theories (where there is no interaction with the matter fields) within the framework of the superfield formalism. In this work [26], we have also provided the reasons behind the uniqueness of the above symmetry transformations and furnished the logical arguments for the non-existence of the on-shell nilpotent anti-BRST symmetry transformations for the 4D non-Abelian theory.

The central theme of our present paper is to generalize the key results of our earlier work in [26] to the more general case of the interacting  $U(1)$  gauge theory where there is an explicit coupling between the 1-form  $U(1)$  gauge field and the Noether conserved current constructed with the matter (Dirac) fields. We find that the GIR on the matter (Dirac) superfields (defined on the above (4, 2)-dimensional supermanifold) enables us to derive the exact nilpotent (anti-)BRST symmetry transformations for the matter (Dirac) fields which can *never* be obtained by exploiting the HC *alone*. The above GIR also provides a meeting-ground for the HC of the usual superfield formalism [1-8] and a gauge invariant condition on the matter superfields.

For our central objective of encapsulating the (anti-)BRST invariance of the 4D Lagrangian density of the interacting  $U(1)$  gauge theory within the framework of the superfield approach to BRST formalism, the following key points are of utmost importance, namely;

- (i) the application of the HC enables us to demonstrate that the kinetic

energy term of the  $U(1)$  gauge field remains independent of the Grassmannian variables when it is expressed in terms of the gauge superfields (obtained after the application of the HC), and

(ii) the application of the above GIR on the matter superfields enables us to show that all the terms containing the matter (Dirac) superfields are independent of the Grassmannian variables when they are expressed in terms of the superfields that are obtained after the application of the HC as well as the GIR on the matter superfields.

The above key restrictions (i.e. HC and GIR) enable us to express the total Lagrangian density of the 4D interacting  $U(1)$  gauge theory with Dirac fields in the language of the superfields, in such a manner that, ultimately, a partial derivative w.r.t.  $\theta$  and/or  $\bar{\theta}$  on it becomes zero. In other words, the total super Lagrangian density (defined on the  $(4, 2)$ -dimensional ‘restricted’ supermanifold) becomes independent of the Grassmannian variables *effectively*. This observation, in turn, implies that the corresponding 4D Lagrangian density of the parent theory (defined on the 4D ordinary spacetime manifold) becomes automatically (anti-)BRST invariant. Stated in the language of geometry on the above ‘restricted’ supermanifold, the translation of the above super Lagrangian density (defined in terms of the superfields obtained after the application of the HC and GIR) along either of the Grassmannian directions ( $\theta$  and/or  $\bar{\theta}$ ) of the above ‘restricted’ supermanifold becomes zero. This is consistent with our earlier observation in [26].

The main motivating factors that have contributed to our curiosity to carry out our present investigation are as follows. First and foremost, it is very important for us to generalize our earlier work [26] (on the geometrical interpretations for the (anti-)BRST invariance in the context of the free 4D (non-)Abelian 1-form gauge theories) to the case where there is an explicit coupling between the  $U(1)$  gauge field and matter (Dirac) fields. Second, it is a challenge to check the validity of the geometrical interpretations provided for the (anti-)BRST invariance in our earlier work [26] to the present case because our present interacting  $U(1)$  gauge theory with Dirac fields is more general than the free case of the 4D (non-)Abelian gauge theories. Finally, our present attempt and earlier works on superfield formalism [10-26] are our modest steps towards our main goal of applying the superfield formalism to the case of higher spin gauge theories that have become popular and pertinent in the modern developments in (super)string theories [27].

Our present paper is organized as follows.

In section 2, we give a brief synopsis of the nilpotent (anti-)BRST symmetry invariance of the Lagrangian density of a 4D interacting  $U(1)$  gauge theory where there is an explicit coupling between the  $U(1)$  gauge field and the Noether conserved current, constructed with the help of the Dirac fields.

We exploit, in section 3, the horizontality condition (HC) (applied to the 1-form super gauge connection defined on the (4, 2)-dimensional supermanifold) to express the kinetic energy, gauge-fixing, and ghost terms in the language of the superfields (derived after the application of HC).

Section 4 deals with a GIR on the matter superfields (defined on the above supermanifold) to obtain the (anti-)BRST symmetry transformations for the matter fields and to express the kinetic energy term, interaction term and mass term of the Dirac fields in the language of the superfields, obtained after the application of the HC and GIR.

Our section 5 is devoted to the derivation of the nilpotent (anti-)BRST symmetry transformations *together* for the gauge, matter and (anti-)ghost fields of the theory from a GIR on the matter superfields and to express the total 4D Lagrangian density in the language of the superfields, obtained after application of the above GIR.

Finally, in section 6, we make some concluding remarks and point out a few future directions for further investigations.

## 2 Nilpotent (Anti-)BRST Symmetry Invariance in QED: A Brief Sketch

We begin with the following nilpotent (anti-)BRST symmetry invariant Lagrangian density for the 4D interacting Abelian 1-form U(1) gauge theory in the Feynman gauge <sup>3</sup> (see, e.g. [9,28,29])

$$\begin{aligned}\mathcal{L}_b &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi + B(\partial \cdot A) + \frac{1}{2}B^2 - i\partial_\mu \bar{C}\partial^\mu C, \\ &\equiv \mathcal{L}_b^{(g)} + \mathcal{L}_b^{(d)}.\end{aligned}\tag{1}$$

In the above, the kinetic energy term for the gauge field is constructed with the help of the curvature tensor  $F_{\mu\nu}$  which is derived from the 2-form  $F^{(2)} = (1/2!)(dx^\mu \wedge dx^\nu)F_{\mu\nu}$ . The latter emerges (i.e.  $F^{(2)} = dA^{(1)}$ ) when the exterior derivative  $d = dx^\mu \partial_\mu$  (with  $d^2 = 0$ ) acts on the 1-form connection  $A^{(1)} = dx^\mu A_\mu$  that defines the gauge potential  $A_\mu$  of our present

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<sup>3</sup>We follow here the convention and notations such that the flat 4D Minkowski spacetime manifold is characterized by a metric  $\eta_{\mu\nu}$  with the signatures  $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$  so that  $\square = \eta_{\mu\nu}\partial^\mu\partial^\nu = (\partial_0)^2 - (\partial_i)^2$  and  $P \cdot Q = \eta_{\mu\nu}P^\mu Q^\nu \equiv P_0Q_0 - P_iQ_i$  is the dot product between two non-null 4-vectors  $P_\mu$  and  $Q_\mu$ . Furthermore, we have  $F_{0i} = E_i, F_{ij} = -\varepsilon_{ijk}B_k$  as the electric and magnetic components of the curvature tensor  $F_{\mu\nu}$ . Here  $\varepsilon_{ijk}$  is the 3D totally antisymmetric Levi-Civita tensor. The Greek indices  $\mu, \nu, \kappa, \dots = 0, 1, 2, 3$  correspond to the spacetime directions and the Latin indices  $i, j, k, \dots = 1, 2, 3$  stand only for the space directions on the above 4D flat spacetime manifold.

theory. The Nakanishi-Lautrup auxiliary vector field  $B$  is invoked to linearize the gauge-fixing term  $[-(1/2)(\partial \cdot A)^2]$ . The latter requires, for the nilpotent (anti-)BRST symmetry invariance in the theory, the fermionic (i.e.  $C^2 = \bar{C}^2 = 0, C\bar{C} + \bar{C}C = 0$ ) (anti-)ghost fields  $(\bar{C})C$  which play central role in the proof of the unitarity of the theory. The covariant derivative  $D_\mu\psi = \partial_\mu\psi + ieA_\mu\psi$  generates the interaction term (i.e.  $-e\bar{\psi}\gamma^\mu A_\mu\psi$ ) between the gauge field  $A_\mu$  and Dirac fields  $\psi$  and  $\bar{\psi}$  with mass  $m$  and charge  $e$ . Here  $\gamma^\mu$ 's are the usual hermitian  $4 \times 4$  Dirac matrices in 4D.

The above Lagrangian (1) has been split into two parts  $\mathcal{L}_b^{(g)}$  and  $\mathcal{L}_b^{(d)}$  for our later convenience. The former corresponds to the kinetic energy term, gauge-fixing term and Faddeev-Popov ghost term (for the 1-form gauge and (anti-)ghost fields) of the theory and the latter corresponds to all the terms in the Lagrangian density that include necessarily the Dirac fields.

The following off-shell nilpotent ( $s_{(a)b}^2 = 0$ ) and anticommuting ( $s_b s_{ab} + s_{ab} s_b = 0$ ) infinitesimal (anti-)BRST symmetry transformations  $s_{(a)b}$ <sup>4</sup>

$$\begin{aligned} s_b A_\mu &= \partial_\mu C, & s_b C &= 0, & s_b \bar{C} &= iB, & s_b B &= 0, \\ s_b \psi &= -ieC\psi, & s_b \bar{\psi} &= -ie\bar{\psi}C, & s_b F_{\mu\nu} &= 0, \end{aligned} \quad (2)$$

$$\begin{aligned} s_{ab} A_\mu &= \partial_\mu \bar{C}, & s_{ab} \bar{C} &= 0, & s_{ab} C &= -iB, & s_{ab} B &= 0, \\ s_{ab} \psi &= -ie\bar{C}\psi, & s_{ab} \bar{\psi} &= -ie\bar{\psi}\bar{C}, & s_{ab} F_{\mu\nu} &= 0, \end{aligned} \quad (3)$$

leave the above Lagrangian density (1) quasi-invariant because it transforms to a total derivative under the above transformations. The key features of the above transformations are:

(i) the curvature tensor, owing its origin to the cohomological operator  $d = dx^\mu \partial_\mu$  (with  $d^2 = 0$ ), remains invariant under both the above nilpotent symmetry transformations (i.e.  $s_{(a)b} F_{\mu\nu} = 0$ ),

(ii) the total terms involving the Dirac fields (i.e.  $\bar{\psi}(i\gamma^\mu D_\mu - m)\psi$ ) also remain invariant under  $s_{(a)b}$  (i.e.  $s_{(a)b}[\bar{\psi}(i\gamma^\mu D_\mu - m)\psi] = 0$ ),

(iii) the nilpotency (i.e.  $d^2 = 0$ ) of the exterior derivative  $d = dx^\mu \partial_\mu$  is replicated in the language of the nilpotency of the above symmetry transformations  $s_{(a)b}$  (i.e.  $s_{(a)b}^2 = 0$ ), and

(iv) there is a deep connection between the exterior derivative and the above nilpotent transformations which will be exploited in the superfield approach to BRST formalism (see, sections 3,4 and 5 below).

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<sup>4</sup>We follow here the notation and convention adopted in [28,29]. In their totality, the (anti-)BRST symmetry transformations  $\delta_{(A)B}$  are the product (i.e.  $\delta_{(A)B} = \eta s_{(a)b}$ ) of an anticommuting ( $\eta C = C\eta, \eta\psi = -\psi\eta$  etc.) spacetime independent parameter  $\eta$  and  $s_{(a)b}$  such that the operator form of the nilpotency of  $\delta_{(A)B}$  is traded with that of  $s_{(a)b}$ .

It can be checked that the gauge-fixing and Faddeev-Popov ghost terms of the Lagrangian density (1) can be written, in the exact form(s), as

$$\begin{aligned}
s_b \left[ -i\bar{C} \{ (\partial \cdot A) + \frac{1}{2} B \} \right] &= B(\partial \cdot A) + \frac{1}{2} B^2 - i\partial_\mu \bar{C} \partial^\mu C, \\
s_{ab} \left[ +iC \{ (\partial \cdot A) + \frac{1}{2} B \} \right] &= B(\partial \cdot A) + \frac{1}{2} B^2 - i\partial_\mu \bar{C} \partial^\mu C, \\
s_b s_{ab} \left( \frac{i}{2} A_\mu A^\mu + \frac{1}{2} C \bar{C} \right) &= B(\partial \cdot A) + \frac{1}{2} B^2 - i\partial_\mu \bar{C} \partial^\mu C,
\end{aligned} \tag{4}$$

modulo some total derivative terms which do not affect the dynamics of the theory. The above expressions provide a simple proof for the nilpotent symmetry invariance of the Lagrangian density (1) because of

- (i) the nilpotency of the transformations  $s_{(a)b}$  (i.e.  $s_{(a)b}^2 = 0$ ),
- (ii) the invariance of the curvature term (i.e.  $s_{(a)b} F_{\mu\nu} = 0$ ) under  $s_{(a)b}$  leading to the invariance of the kinetic energy term for the gauge field, and
- (iii) the invariance of the terms involving Dirac fields (i.e.  $s_{(a)b} [\bar{\psi}(i\gamma^\mu D_\mu - m)\psi] = 0$ ) under the nilpotent (anti-)BRST symmetry transformations  $s_{(a)b}$ .

### 3 (Anti-)BRST Transformations for the Gauge and (Anti-)ghost Fields: Horizontality Condition

We exploit here the usual superfield approach [1-9] to obtain the nilpotent (anti-)BRST symmetry transformations for the gauge and (anti-)ghost fields of the Lagrangian density (1). To this end in mind, first of all, we generalize the 4D local fields  $(A_\mu(x), C(x), \bar{C}(x))$  to the superfields  $(\mathcal{B}_\mu(x, \theta, \bar{\theta}), \mathcal{F}(x, \theta, \bar{\theta}), \bar{\mathcal{F}}(x, \theta, \bar{\theta}))$  that are defined on a  $(4, 2)$ -dimensional supermanifold, parametrized by the superspace variable  $Z^M = (x^\mu, \theta, \bar{\theta})$ . In terms of these superfields, we can define a super 1-form connection as

$$\tilde{A}^{(1)} = dZ^M \tilde{A}_M = dx^\mu \mathcal{B}_\mu + d\theta \bar{\mathcal{F}} + d\bar{\theta} \mathcal{F}, \tag{5}$$

where  $\tilde{A}_M$  is the vector superfield with the component multiplet fields as  $(\mathcal{B}_\mu, \mathcal{F}, \bar{\mathcal{F}})$  and  $dZ^M = (dx^\mu, d\theta, d\bar{\theta})$  is the superspace differential.

The above component superfields can be expanded in terms of the basic fields  $(A_\mu, C, \bar{C})$ , auxiliary field  $B$  and secondary fields as (see, e.g. [4-7])

$$\begin{aligned}
\mathcal{B}_\mu(x, \theta, \bar{\theta}) &= A_\mu(x) + \theta \bar{R}_\mu(x) + \bar{\theta} R_\mu(x) + i\theta \bar{\theta} S_\mu(x), \\
\mathcal{F}(x, \theta, \bar{\theta}) &= C(x) + i\theta \bar{B}(x) + i\bar{\theta} B(x) + i\theta \bar{\theta} s(x), \\
\bar{\mathcal{F}}(x, \theta, \bar{\theta}) &= \bar{C}(x) + i\theta \bar{B}(x) + i\bar{\theta} B(x) + i\theta \bar{\theta} \bar{s}(x),
\end{aligned} \tag{6}$$

where the secondary fields are  $\bar{B}(x), \mathcal{B}(x), \bar{\mathcal{B}}(x), s(x), \bar{s}(x)$ . It will be noted that, in the limit  $(\theta, \bar{\theta}) \rightarrow 0$ , we retrieve our basic fields  $(A_\mu, C, \bar{C})$ . In the above expansion, the bosonic and fermionic component fields do match with each-other. The exact expressions for the secondary fields can be derived in terms of the basic fields of the theory if we exploit the celebrated HC.

Let us recall our observation after the (anti-)BRST symmetry transformations (2) and (3). These transformations  $s_{(a)b}$  owe their origin to the exterior derivative  $d = dx^\mu \partial_\mu$  which plays a very important role in the application of the HC. To this end in mind, let us generalize the 4D ordinary  $d$  to its counterpart on the (4, 2)-dimensional supermanifold, as

$$d \rightarrow \tilde{d} = dZ^M \partial_M = dx^\mu \partial_\mu + d\theta \partial_\theta + d\bar{\theta} \partial_{\bar{\theta}}, \quad \partial_M = (\partial_\mu, \partial_\theta, \partial_{\bar{\theta}}). \quad (7)$$

The HC is the requirement that the super 2-form  $\tilde{F}^{(2)} = \tilde{d}\tilde{A}^{(1)}$ , defined on the (4, 2)-dimensional supermanifold, should be equal (i.e.  $\tilde{F}^{(2)} = F^{(2)}$ ) to the ordinary 4D 2-form  $F^{(2)} = dA^{(1)}$ . Physically, this amounts to the restriction that the gauge (i.e. (anti-)BRST) invariant quantities  $E_i$  and  $B_i$ , which are the components of the curvature tensor  $F_{\mu\nu}$ , should remain *unaffected* by the presence of the Grassmannian variables  $\theta$  and  $\bar{\theta}$ .

The explicit computations for  $\tilde{F}^{(2)}$  (from (5) and (7)) and its subsequent equality with the ordinary 4D 2-form  $F^{(2)}$ , leads to the following relationships between the secondary fields and basic fields of the theory [21-25]

$$\begin{aligned} R_\mu &= \partial_\mu C, \quad \bar{R}_\mu = \partial_\mu \bar{C}, \quad S_\mu = \partial_\mu B \equiv -\partial_\mu \bar{B}, \\ B + \bar{B} &= 0, \quad \mathcal{B} = 0, \quad \bar{\mathcal{B}} = 0, \quad s = 0, \quad \bar{s} = 0. \end{aligned} \quad (8)$$

Insertion of these values into the expansion (6) of the superfields leads to the following final expansion

$$\begin{aligned} \mathcal{B}_\mu^{(h)}(x, \theta, \bar{\theta}) &= A_\mu(x) + \theta \partial_\mu \bar{C}(x) + \bar{\theta} \partial_\mu C(x) + i \theta \bar{\theta} \partial_\mu B(x) \\ &\equiv A_\mu(x) + \theta(s_{ab} A_\mu(x)) + \bar{\theta}(s_b A_\mu(x)) + \theta \bar{\theta} (s_b s_{ab} A_\mu(x)), \\ \mathcal{F}^{(h)}(x, \theta, \bar{\theta}) &= C(x) - i \theta B(x) \\ &\equiv C(x) + \theta (s_{ab} C(x)) + \bar{\theta} (s_b C(x)) + \theta \bar{\theta} (s_b s_{ab} C(x)), \\ \bar{\mathcal{F}}^{(h)}(x, \theta, \bar{\theta}) &= \bar{C}(x) + i \bar{\theta} B(x) \\ &\equiv C(x) + \theta (s_{ab} C(x)) + \bar{\theta} (s_b C(x)) + \theta \bar{\theta} (s_b s_{ab} C(x)), \end{aligned} \quad (9)$$

where the following points are important, namely;

(i) the superscript  $(h)$  on the superfields corresponds to the superfields obtained after the application of the HC,

(ii) the transformations  $s_b C = 0$  and  $s_{ab} \bar{C} = 0$  have been taken into account in the above uniform expansions,



(iii) the super curvature tensor  $\tilde{F}_{\mu\nu}^{(h)} = \partial_\mu \mathcal{B}_\nu^{(h)} - \partial_\nu \mathcal{B}_\mu^{(h)}$  is found to be equal to the ordinary curvature tensor  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  because all the Grassmannian dependent terms cancel out to become zero, and

(iv) there is a very intimate relationship between the (anti-)BRST symmetry transformations acting on a 4D field and the translational generators (along the Grassmannian directions of the supermanifold) acting on the corresponding (4, 2)-dimensional superfield (obtained after the application HC). Mathematically, this statement can be expressed as

$$\text{Lim}_{\theta \rightarrow 0} \frac{\partial}{\partial \theta} \tilde{\Omega}^{(h)}(x, \theta, \bar{\theta}) = s_b \Omega(x), \quad \text{Lim}_{\bar{\theta} \rightarrow 0} \frac{\partial}{\partial \bar{\theta}} \tilde{\Omega}^{(h)}(x, \theta, \bar{\theta}) = s_{ab} \Omega(x), \quad (10)$$

where  $\Omega(x)$  is the ordinary 4D generic local field and  $\tilde{\Omega}^{(h)}(x, \theta, \bar{\theta})$  is the corresponding generic superfield derived after the application of the HC.

The above superfields would be used to express the kinetic energy term for the U(1) gauge field, gauge-fixing term for the photon field and Faddeev-Popov ghost terms for the (anti-)ghost fields of the theory. These can be expressed in the following *three* different and distinct forms (see, e.g. [26])

$$\begin{aligned} \tilde{\mathcal{L}}_{(g)}^{(1)} &= -\frac{1}{4} \tilde{F}_{\mu\nu}^{(h)} \tilde{F}^{\mu\nu(h)} + \text{Lim}_{\theta \rightarrow 0} \frac{\partial}{\partial \theta} \left[ -i \bar{\mathcal{F}}^{(h)} (\partial^\mu \mathcal{B}_\mu^{(h)} + \frac{1}{2} B) \right], \\ \tilde{\mathcal{L}}_{(g)}^{(2)} &= -\frac{1}{4} \tilde{F}_{\mu\nu}^{(h)} \tilde{F}^{\mu\nu(h)} + \text{Lim}_{\bar{\theta} \rightarrow 0} \frac{\partial}{\partial \bar{\theta}} \left[ +i \mathcal{F}^{(h)} (\partial^\mu \mathcal{B}_\mu^{(h)} + \frac{1}{2} B) \right], \\ \tilde{\mathcal{L}}_{(g)}^{(3)} &= -\frac{1}{4} \tilde{F}_{\mu\nu}^{(h)} \tilde{F}^{\mu\nu(h)} + \frac{\partial}{\partial \theta} \frac{\partial}{\partial \bar{\theta}} \left[ +\frac{i}{2} \mathcal{B}^{\mu(h)} \mathcal{B}_\mu^{(h)} + \frac{1}{2} \mathcal{F}^{(h)} \bar{\mathcal{F}}^{(h)} \right], \end{aligned} \quad (11)$$

where the subscript (g) on the above super Lagrangians stands for the terms in the 4D Lagrangian density that correspond to the ‘gauge’ and ‘ghost’ fields. In other words, we have encapsulated the kinetic energy term, gauge-fixing term and Faddeev-Popov ghost term of the 4D Lagrangian density (1) in the language of the superfields derived after the application of the HC.

It is very interesting to note that the (anti-)BRST invariance of the kinetic energy term, gauge-fixing term and Faddeev-Popov ghost term can be captured in the language of the translations of the above super Lagrangian densities  $\tilde{\mathcal{L}}_{(g)}^{(1,2,3)}$  along the Grassmannian directions. Mathematically, the nilpotent BRST invariance can be expressed as follows

$$s_b \mathcal{L}_b^{(g)} = 0 \Leftrightarrow \text{Lim}_{\theta \rightarrow 0} \frac{\partial}{\partial \theta} \tilde{\mathcal{L}}_{(g)}^{(1)} = 0, \quad s_b^2 = 0 \Leftrightarrow (\text{Lim}_{\theta \rightarrow 0} \frac{\partial}{\partial \theta})^2 = 0, \quad (12)$$

$$s_b \mathcal{L}_b^{(g)} = 0 \Leftrightarrow \frac{\partial}{\partial \theta} \tilde{\mathcal{L}}_{(g)}^{(3)} = 0, \quad s_b^2 = 0 \Leftrightarrow (\frac{\partial}{\partial \theta})^2 = 0. \quad (13)$$

Similarly, the anti-BRST invariance of the kinetic energy, gauge-fixing and ghost terms can be expressed, in the language of the superfields, as follows

$$s_{ab}\mathcal{L}_b^{(g)} = 0 \Leftrightarrow \text{Lim}_{\bar{\theta} \rightarrow 0} \frac{\partial}{\partial \theta} \tilde{\mathcal{L}}_{(g)}^{(2)} = 0, \quad s_{ab}^2 = 0 \Leftrightarrow (\text{Lim}_{\bar{\theta} \rightarrow 0} \frac{\partial}{\partial \theta})^2 = 0, \quad (14)$$

$$s_{ab}\mathcal{L}_b^{(g)} = 0 \Leftrightarrow \frac{\partial}{\partial \theta} \tilde{\mathcal{L}}_{(g)}^{(3)} = 0, \quad s_{ab}^2 = 0 \Leftrightarrow (\frac{\partial}{\partial \theta})^2 = 0. \quad (15)$$

Thus, we note that the Grassmannian independence of the super Lagrangian densities, defined in terms of the superfields (derived after the application of the HC) on the (4, 2)-dimensional supermanifold, automatically implies the (anti-)BRST invariance of the 4D Lagrangian density defined in terms of 4D local fields taking their values on a 4D flat Minkowskian spacetime manifold.

## 4 (Anti-)BRST Symmetry Transformations for Dirac Fields: Gauge Invariant Condition

Unlike the superfields  $\mathcal{B}_\mu(x, \theta, \bar{\theta})$ ,  $\mathcal{F}(x, \theta, \bar{\theta})$ ,  $\bar{\mathcal{F}}(x, \theta, \bar{\theta})$  that form a vector supermultiplet (cf. previous section), the fermionic matter superfields  $\Psi(x, \theta, \bar{\theta})$  and  $\bar{\Psi}(x, \theta, \bar{\theta})$  (which are the generalizations of the 4D matter Dirac fields  $\psi(x)$  and  $\bar{\psi}(x)$  of the Lagrangian density (1) on the (4, 2)-dimensional supermanifold) do not belong to any supermultiplet. Thus, it appears that there is no connection between the superfields  $(\mathcal{B}_\mu, \mathcal{F}, \bar{\mathcal{F}})$  and the matter superfields  $(\Psi(x, \theta, \bar{\theta}), \bar{\Psi}(x, \theta, \bar{\theta}))$ . However, there is one *gauge invariant* relationship in which the matter superfields do ‘talk’ with the super gauge connection  $\tilde{A}^{(1)}$  of (5). We exploit this relationship and impose this GIR on the matter superfields, defined on the (4, 2)-dimensional supermanifold. The above GIR on the matter (super) fields is as follows [16,17]

$$\bar{\Psi}(x, \theta, \bar{\theta}) [\tilde{d} + ie\tilde{A}_{(h)}^{(1)}] \Psi(x, \theta, \bar{\theta}) = \bar{\psi}(x)(d + ieA^{(1)})\psi(x). \quad (16)$$

It can be easily verified that the r.h.s. of the above equation is a U(1) gauge (i.e. (anti-)BRST) invariant quantity because it is connected with the covariant derivative. Written in the explicit form, the r.h.s. is  $dx^\mu \bar{\psi}(x) D_\mu \psi(x)$  where  $d = dx^\mu \partial_\mu$  and  $A^{(1)} = dx^\mu A_\mu$  have been taken into account in the definition of the covariant 1-form derivative  $D = dx^\mu D_\mu \equiv d + ieA^{(1)}$ .

The relationship (16) is interesting on the following grounds. First, it is a gauge (i.e. (anti-)BRST) invariant quantity. Thus, it is physical in some sense. Second, it will be noted that, on the l.h.s. of (16), we have  $\tilde{A}_{(h)}^{(1)}$  which

is derived after the application of HC<sup>5</sup>. Thus, HC of the previous section and GIR of our present section (cf. (16)) are intimately connected. In fact, the explicit form of  $\tilde{A}_{(h)}^{(1)} = dx^\mu \mathcal{B}_\mu^{(h)} + d\theta \bar{\mathcal{F}}^{(h)} + d\bar{\theta} \mathcal{F}^{(h)}$  is such that the whole expansion of (9) is going to play a very decisive role in the determination of the exact nilpotent (anti-)BRST symmetry transformations for the matter fields in the language of the (anti-)ghost fields and matter fields themselves.

To find out the impact of the above restriction, we have to expand the matter superfields along the Grassmannian  $\theta$  and  $\bar{\theta}$  directions of the (4, 2)-dimensional supermanifold as follows

$$\begin{aligned}\Psi(x, \theta, \bar{\theta}) &= \psi(x) + i\theta \bar{b}_1(x) + i\bar{\theta} b_2(x) + i\theta \bar{\theta} f(x), \\ \bar{\Psi}(x, \theta, \bar{\theta}) &= \bar{\psi}(x) + i\theta \bar{b}_2(x) + i\bar{\theta} b_1(x) + i\theta \bar{\theta} \bar{f}(x),\end{aligned}\quad (17)$$

where  $\psi(x)$  and  $\bar{\psi}(x)$  are the 4D basic Dirac fields of the Lagrangian density (1) and  $b_1, \bar{b}_1, b_2, \bar{b}_2, f, \bar{f}$  are the secondary fields which will be expressed in terms of the basic fields of the Lagrangian density (1) due to the above GIR (16). In the limit  $(\theta, \bar{\theta}) \rightarrow 0$ , we retrieve our basic 4D local fields  $\psi$  and  $\bar{\psi}$  and bosonic ( $b_1, b_2, \bar{b}_1, \bar{b}_2$ ) and fermionic ( $\psi, \bar{\psi}, f, \bar{f}$ ) degrees of freedom do match in the above expansion. This is consistent with the basic requirements of a true supersymmetric field theory.

The explicit computation of the l.h.s. of GIR (16) and its subsequent comparison with the r.h.s., yields the following relationship between the secondary fields and the basic fields of the theory (see [17] for details)

$$\begin{aligned}b_1 &= -e\bar{\psi}C, & b_2 &= -eC\psi, & \bar{b}_1 &= -e\bar{C}\psi, & \bar{b}_2 &= -e\bar{\psi}\bar{C}, \\ f &= -ie[B + e\bar{C}C]\psi, & \bar{f} &= +ie\bar{\psi}[B + eC\bar{C}].\end{aligned}\quad (18)$$

In the above explicit computation, all the expansions (9) (obtained after the application of the HC) have been used. The insertions of the above values into the expansion of the matter superfields (17), lead to

$$\begin{aligned}\Psi^{(G)}(x, \theta, \bar{\theta}) &= \psi(x) + \theta(-ie\bar{C}\psi(x)) + \bar{\theta}(-ieC\psi(x)) \\ &\quad + \theta\bar{\theta}[e(B + e\bar{C}C)\psi(x)], \\ &\equiv \psi(x) + \theta(s_{ab}\psi(x)) + \bar{\theta}(s_b\psi(x)) + \theta\bar{\theta}(s_b s_{ab}\psi(x)), \\ \bar{\Psi}^{(G)}(x, \theta, \bar{\theta}) &= \bar{\psi}(x) + \theta(-ie\bar{\psi}(x)\bar{C}) + \bar{\theta}(-ie\bar{\psi}(x)C) \\ &\quad + \theta\bar{\theta}[-e\bar{\psi}(x)(B + eC\bar{C})], \\ &\equiv \bar{\psi}(x) + \theta(s_{ab}\bar{\psi}(x)) + \bar{\theta}(s_b\bar{\psi}(x)) + \theta\bar{\theta}(s_b s_{ab}\bar{\psi}(x)).\end{aligned}\quad (19)$$

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<sup>5</sup>The HC is basically a gauge *covariant* restriction for the discussion of the non-Abelian gauge theory. It, however, reduces to a GIR for the Abelian U(1) gauge theory.

The superscript  $(G)$  on the above superfields denotes the fact that they have been obtained after the application of the GIR (cf. equation (16)) on the matter superfields (defined on the  $(4, 2)$ -dimensional supermanifold).

It is very interesting to note, at this stage, that the GIR (cf. (16)) on the matter superfields leads to

(i) the exact and unique derivation of the nilpotent (anti-)BRST symmetry transformations for the matter fields  $\psi(x)$  and  $\bar{\psi}(x)$ , and

(ii) the geometrical interpretation for the (anti-)BRST symmetry transformations as the translational generators along the Grassmannian directions  $\theta$  and  $\bar{\theta}$  of the above supermanifold. As a consequence, we obtain the analogue of the equation (10), as

$$\text{Lim}_{\theta \rightarrow 0} \frac{\partial}{\partial \bar{\theta}} \tilde{\Omega}^{(G)}(x, \theta, \bar{\theta}) = s_b \Omega(x), \quad \text{Lim}_{\bar{\theta} \rightarrow 0} \frac{\partial}{\partial \theta} \tilde{\Omega}^{(G)}(x, \theta, \bar{\theta}) = s_{ab} \Omega(x), \quad (20)$$

where  $\tilde{\Omega}^{(G)}(x, \theta, \bar{\theta})$  stands for the super expansions (19) for the matter superfields, obtained after the application of the GIR (16). The generic field  $\Omega(x)$  stands for the 4D matter Dirac fields  $\psi(x)$  and  $\bar{\psi}(x)$ .

It is self-evident that we have obtained super expansion in (19) due to the GIR given in (16). There is an interesting consequence due to our expansion in (19). It is straightforward to check that the following equation

$$\bar{\Psi}^{(G)}(x, \theta, \bar{\theta}) \Psi^{(G)}(x, \theta, \bar{\theta}) = \bar{\psi}(x) \psi(x), \quad (21)$$

is automatically satisfied. This observation implies, ultimately, that the equation (16) can be re-expressed as

$$\bar{\Psi}^{(G)}(x, \theta, \bar{\theta}) [\tilde{d} + ie\tilde{A}_{(h)}^{(1)} - m] \Psi^{(G)}(x, \theta, \bar{\theta}) = \bar{\psi}(x)(d + ieA^{(1)} - m)\psi(x). \quad (22)$$

Physically, the above restriction implies that the total Dirac part of the Lagrangian density (1) (i.e.  $\mathcal{L}^{(d)}$ ) remain unaffected due to the presence of the Grassmannian variables on the  $(4, 2)$ -dimensional supermanifold, on which, our 4D interacting  $U(1)$  gauge theory (with Dirac fields) has been considered. The above observation implies that the super Lagrangian density for the Dirac fields, using GIR (16) and super expansion (19), can be written as

$$\tilde{\mathcal{L}}^{(d)} = \bar{\Psi}^{(G)}(x, \theta, \bar{\theta})(i\gamma^M D_M^{(h)} - m) \Psi^{(G)}(x, \theta, \bar{\theta}) \equiv \mathcal{L}^{(d)}, \quad (23)$$

where  $\gamma^M$ 's are some non-trivial generalization of the  $4 \times 4$  Dirac matrices  $\gamma^\mu$  to the  $(4, 2)$ -dimensional supermanifold and  $\gamma^M D_M^{(h)}$  is defined as<sup>6</sup>

$$\gamma^M D_M^{(h)} = \gamma^\mu (\partial_\mu + ie\mathcal{B}_\mu^{(h)}) + (\partial_\theta + ie\bar{\mathcal{F}}^{(h)}) + (\partial_{\bar{\theta}} + ie\mathcal{F}^{(h)}). \quad (24)$$

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<sup>6</sup>It can be readily checked that the  $(4, 2)$ -dimensional representation  $\gamma^M = \text{diag}(\gamma^\mu, 1, 1)$  serves our purpose in equation (24). In general, one can choose, however, another representation  $\gamma^M = \text{diag}(\gamma^\mu, C_1, C_2)$  without altering our main results even if the constants  $C_1$  and  $C_2$  (with  $C_1 \neq C_2$ ) are chosen to be different from one.

Here the superfields  $\mathcal{B}_\mu^{(h)}$ ,  $\mathcal{F}^{(h)}$  and  $\bar{\mathcal{F}}^{(h)}$  are the expanded form of equation (9) that have been obtained after the application of HC.

It is straightforward to capture now the (anti-)BRST invariance (i.e.  $s_{(a)b} [\bar{\psi}(i\gamma^\mu D_\mu - m)\psi] = 0$ ) of the Dirac part of the Lagrangian density (1) in the language of the superfields. This can be expressed as follows

$$\begin{aligned} s_b \mathcal{L}^{(d)} &= 0 \Leftrightarrow \text{Lim}_{\theta \rightarrow 0} \frac{\partial}{\partial \theta} \tilde{\mathcal{L}}^{(d)} = 0, \\ s_{ab} \mathcal{L}^{(d)} &= 0 \Leftrightarrow \text{Lim}_{\bar{\theta} \rightarrow 0} \frac{\partial}{\partial \bar{\theta}} \tilde{\mathcal{L}}^{(d)} = 0, \end{aligned} \quad (25)$$

In the language of the geometry on the (4, 2)-dimensional supermanifold, the (anti-)BRST invariance of the 4D Lagrangian density  $\mathcal{L}^{(d)}$  is equivalent to the translations of some composite superfields (present in the super Lagrangian density  $\tilde{\mathcal{L}}^{(d)}$ ) along the Grassmannian directions ( $\theta$  and/or  $\bar{\theta}$ ) of the supermanifold such that the outcome of the translation is *zero*.

## 5 (Anti-)BRST Transformations for All the Fields: Single Gauge Invariant Restriction

In this section, we recapitulate some key points connected with the derivation of the results of sections 3 and 4 from a single GIR on the superfields, defined on the (4, 2)-dimensional supermanifold [18]. This GIR also owes its origin to the (super) covariant derivatives but, in a form, that is quite different from (16). The explicit form of this GIR is<sup>7</sup>

$$\bar{\Psi}(x, \theta, \bar{\theta}) \tilde{D} \tilde{D} \Psi(x, \theta, \bar{\theta}) = \bar{\psi}(x) D D \psi(x), \quad (26)$$

where  $\tilde{D}$  and  $D$  are the covariant derivatives defined on the (4, 2)-dimensional supermanifold and 4D flat Minkowski spacetime manifold, respectively. These are defined as follows

$$\tilde{D} = \tilde{d} + ie\tilde{A}^{(1)}, \quad D = d + ieA^{(1)}, \quad d = dx^\mu \partial_\mu, \quad A^{(1)} = dx^\mu A_\mu, \quad (27)$$

where all the quantities, in the above, have been taken from the earlier sections 3 and 4. For instance, equations (5), (7) and (17) have been used.

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<sup>7</sup> The stunning strength of this equation does not appear in the context of the Abelian U(1) gauge theory because all the fields are commutative in nature. Its immense mathematical power gets reflected in its full blaze of glory in the non-Abelian gauge theory where it leads to the *exact* derivation of all the nilpotent (anti-)BRST symmetry transformations for all the fields of the 4D non-Abelian gauge theory [19]. The non-Abelian fields are, in general, the noncommutative fields that are Lie algebraic valued.

The above restriction is a GIR on the superfields (defined on the above supermanifold) because of the fact that the r.h.s. of (26) is

$$\bar{\psi}DD\psi = \frac{ie}{2}(dx^\mu \wedge dx^\nu)\bar{\psi}F_{\mu\nu}\psi \equiv ie\bar{\psi}F^{(2)}\psi. \quad (28)$$

It is straightforward to check that the above quantity is U(1) gauge invariant. A noteworthy point, at this stage, is that the r.h.s. is a 2-form with the differentials (i.e.  $dx^\mu \wedge dx^\nu$ ) in the spacetime variables *only*. However, the l.h.s. of the equation (26) contains all the differential 2-forms in terms of the superspace variables. It is obvious that all the coefficients of the differentials in the Grassmannian variables of the l.h.s. will be set equal to zero because of the equality in (26), in which, there are no Grassmannian variables on the r.h.s. The latter is explicitly expressed in (28).

It has been clearly demonstrated in our earlier work [18] that the outcomes of the above equality in (26) (i.e. GIR) are the relationships that we have already obtained separately and independently in (8) and (18). Thus, the expansions of the superfields, ultimately, reduce to the forms which are given in (9) and (19). As a consequence, we obtain all the (anti-)BRST symmetry transformations for all the fields of the 4D theory, described by the Lagrangian density (1). It is important to note that the geometrical interpretations of the (anti-)BRST symmetry transformations (and their corresponding generators) as the translational generators (along the Grassmannian directions of the above (4, 2)-dimensional supermanifold) remain unchanged even under the restriction (26).

It is obvious, from our above discussions, that the total (anti-)BRST invariant Lagrangian density (1), defined in terms of the local fields (taking their values on the 4D flat spacetime manifold), can be recast in terms of the superfields (defined on the (4, 2)-dimensional supermanifold) by adding super Lagrangian densities given in equations (11) and (23). This can be mathematically expressed as

$$\tilde{\mathcal{L}}_T = \tilde{\mathcal{L}}_{(g)}^{(1,2,3)} + \tilde{\mathcal{L}}^{(d)}. \quad (29)$$

Now the nilpotent (anti-)BRST invariance of the Lagrangian density (1) can be expressed, in the language of the superfields (obtained after HC and GIR) and the translational generators along the Grassmannian directions of the above supermanifold as (25) by the replacement:  $\tilde{\mathcal{L}}^{(d)} \rightarrow \tilde{\mathcal{L}}_T$ . Finally, we conclude that the Grassmannian independence of the super Lagrangian density encodes the (anti-)BRST invariance of the 4D Lagrangian density.

Explained in other words, if the partial derivative w.r.t.  $\bar{\theta}$  of a super Lagrangian density is zero, the corresponding 4D Lagrangian density will be

BRST invariant. In a similar fashion, the anti-BRST invariance can be explained in terms of the partial derivative w.r.t.  $\theta$ . Such (anti-)BRST invariant super Lagrangians can also be expressed in terms of the partial derivatives w.r.t. *both* the Grassmannian variables  $\theta$  and  $\bar{\theta}$ . Geometrically, whenever the translation of the super Lagrangian densities (expressed in terms of the composite (super)fields), along either of the Grassmannian directions is zero, the corresponding 4D Lagrangian density will have (anti-)BRST invariance.

## 6 Conclusions

In our present endeavour, we have focussed on the (anti-)BRST invariance of the Lagrangian density of a 4D interacting U(1) gauge theory with Dirac fields. As in our earlier work [26] on the free 4D (non-)Abelian 1-form gauge theories (without having any kind of interactions with the matter fields), we find that the Grassmannian independence of the super Lagrangian densities (that are expressed in terms of the superfields obtained after the application of the HC and GIR) is a sure guarantee that the corresponding 4D Lagrangian density would respect the nilpotent (anti-)BRST symmetry invariance.

In the language of the geometry on the  $(4, 2)$ -dimensional supermanifold (on which our present 4D interacting U(1) gauge theory with Dirac fields has been considered) if the translation of the super Lagrangian densities along

(i) the  $\bar{\theta}$ -direction of the above supermanifold (without any translation along the  $\theta$ -direction) is zero, the corresponding 4D Lagrangian density would possess the nilpotent BRST invariance,

(ii) the  $\theta$ -direction of the above supermanifold (without any shift along the  $\bar{\theta}$ -direction) is zero, there will be nilpotent anti-BRST invariance for the 4D Lagrangian density of the theory, and

(iii) the  $\theta$ - and  $\bar{\theta}$ -directions (one followed by the other; either ways) is zero, there would be existence of the (anti-)BRST symmetry invariance *together* for the 4D Lagrangian density of the theory.

We have been able to show in our present work (as well as in our earlier works [10-26]) that the nilpotent internal (anti-)BRST symmetry transformations  $s_{(a)b}$  for the 4D theories are very intimately connected with the translational generators  $(\partial_\theta, \partial_{\bar{\theta}})$  along the Grassmannian directions of the  $(4, 2)$ -dimensional supermanifold. Thus, one of the key features of our superfield approach to BRST formalism is the sure guarantee that the nilpotency (i.e.  $s_{(a)b}^2 = 0$ ) and the anticommutativity (ie.  $s_b s_{ab} + s_{ab} s_b = 0$ ) are the two key *basic* properties of the (anti-)BRST symmetry transformations  $s_{(a)b}$ . These salient features would *always* be present in the BRST approach to any arbitrary  $p$ -form ( $p = 1, 2, 3, \dots$ ) gauge theory which is endowed with the

first-class constraints (Dirac's prescription) [30,31]. In other words, these specific features are the integral ingredients of our superfield approach to BRST formalism because the translational generators  $(\partial_\theta, \partial_{\bar\theta})$  always obey the nilpotency property  $(\partial_\theta^2 = 0, \partial_{\bar\theta}^2 = 0)$  as well as the anticommutativity property (i.e.  $\partial_\theta \partial_{\bar\theta} + \partial_{\bar\theta} \partial_\theta = 0$ ) due to the basic anticommuting nature of the Grassmannian variables  $\theta$  and  $\bar\theta$ .

We would like to make some remarks on our GIRs on the matter superfields (cf. (16) and (26)) which lead to the derivation of

- (i) the nilpotent (anti-BRST) symmetry transformations for the matter (Dirac) fields (cf. (16)) of our present interacting U(1) gauge theory, and
- (ii) the nilpotent (anti-)BRST symmetry transformations of all the fields of the our 4D interacting U(1) 1-form gauge theory.

One of the key ingredient in these restrictions, on the matter superfields of the (4, 2)-dimensional supermanifold, is the idea of gauge invariance. It will be noted that the covariant version of these restrictions lead to completely absurd results (see, e.g. [17] for details).

The important “geometrical” consequences of these GIRs on the matter superfields (cf. (16) and (26)) are as follows

- (i) the Grassmannian independence of the Dirac part of the super Lagrangian density  $(\tilde{\mathcal{L}}^{(d)})$  due to the application of (16), and
- (ii) the Grassmannian independence of the kinetic energy term for the U(1) gauge superfield and the super Lagrangian density  $\tilde{\mathcal{L}}^{(d)}$  due to the application of the GIR (26).

In fact, it is worthwhile to mention that the GIR (26) on the matter superfields provides a generalization of the HC because it leads to the results that are obtained due to the application of the HC and GIR (16) separately.

One of the highlights of our present investigation is the simplicity and beauty that have been brought out for the (anti-)BRST invariance of the Lagrangian density of the 4D interacting U(1) gauge theory within the framework of the superfield approach to BRST formalism. It would be nice to generalize our present work to the case of the interacting 4D non-Abelian gauge theory (with Dirac fields) because the latter theory is more general than our present interacting Abelian U(1) gauge theory. We have also devoted time on the nilpotent (anti-)BRST as well as the nilpotent (anti-)co-BRST invariance of the 4D free Abelian 2-form gauge theory in the Lagrangian formalism [32]. It would be interesting endeavour to capture the nilpotent (anti-)BRST (as well as (anti-)co-BRST) invariance of the 4D (non-)Abelian 2-form and higher spin gauge theories in the framework of the superfield approach to BRST formalism. These are some of the issues that are under investigation at the moment and our results would be reported elsewhere [33].



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